Title of the Research:

A study on completely Archimedean ordered semigroups

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SYNOPSIS

Key Words : ordered semigroup, ordered inverse, ordered idempotent, regular , completely regular, completely π-regular, Archimedean, t-Archimedean, simple, semilattice decomposition, nil-extension, chain decomposition.

1 Introduction

In 1941, A. H. Clifford[1] characterized the semigroups which are union of groups and the semigroups which are semilattices of groups. Later, it has been proved that the technique of decomposing a semigroup into equivalence classes which are again semigroups and of a better known, indeed of simpler structure, is most efficient to describe the structure of different particular classes of semigroups. Presently the semigroups which are union of groups are called completely regular semigroups and which are semilattices of groups are called Clifford semigroups.

In ordered semigroup (with a compatible partial order) there are vast scope to study completely Archimedeaness. Despite a huge scope of work these ordered semigroups have not been yet studied due to the complexity of generalization.

Here our main aim is to introduce the notion of completely Archimedean ordered semigroup and to characterize their complete semilattice and chain decompositions.

2 Brief History of Ordered Semigroup:

An ordered semigroup(po-semigroup) $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ together a partial order $\leq$ on S such that for all $a, b, c \in S$, $a \leq b$ implies that $ac \leq bc$ and $ca \leq cb$. An idempotent of S is an element e of S such that $e \leq e^2$. A band is a semigroup in which every element is an idempotent. A commutative band is called a semilattice. The identity, if it exists, of a semigroup S is denoted by 1. If S has no identity element then it is very easy to adjoin an extra element 1 to S to form a monoid, in the following way:

$$1s = s1 = s, \text{ for all } s \in S \text{ and } 11 = 1.$$

We then define $S^1 = S$ if S has the identity element, and $S^1 = S \cup \{1\}$ otherwise.

Let T be a subsemigroup of S and let H be a nonempty subset of T. We denote: $(H]_T = \{x \in T \mid (\exists h \in H)\ x \leq h\ \}$. If $T = S$, then $(H]_S$ is denoted simply by $(H]$. We have $H \subseteq (H]_T \subseteq (H]$ *and* $A \subseteq B \Rightarrow\ (A]_T \subseteq\ (B]_T$, for any nonempty subsets A, B of T. For $A, B \subseteq S$, define $AB = \{ab \mid a \in A,\ b \in B\}$. A nonempty subset I of S is called a left (right) ideal of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] \subseteq I$. I is an ideal of S if it is both a left and a right ideal of S. A subsemigroup B of S is called a bi-ideal if $BSB \subseteq B$ and $(B] \subseteq B$.

An ordered semigroup S is called simple (left, right) if it does not have any proper ideal (left, right, bi-). An ordered semigroup which is both left and right simple is called t-simple. The t-simple subsemigroups generalize the notion of subgroups in a semigroup without order. This is not at all surprising, since a semigroup (without order) is a group if and only if it is both a left simple and right simple semigroup.

Since intersection of any collection of ideals (left, right, bi-) is an ideal (left, right, bi-), there is the smallest ideal (left, right, bi-) containing $a \in S$ which we denote by $I(a)$ $(L(a), R(a), B(a))$ respectively, and call as the ideal (left, right, bi-) generated by a. Analogous to the Green's relations on a semigroup without order, Kehayopulu introduced four equivalence relations on an ordered semigroup as $a\mathcal{L}b\ if\ L(a) = L(b)$, $a\mathcal{R}b\ if\ R(a) = R(b)$, $a\mathcal{J}b\ if\ I(a) = I(b)$ *and* $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

Also the equivalence relation $\mathcal{H}_0$ defined by: $a\mathcal{H}_0 b\ if\ B(a) = B(b)$ plays an important role to characterize the structure of ordered semigroups. An ordered semigroup S is called a regular (left regular, right regular) semigroup if for every $a \in S$, $a \in (aSa]$ $(a \in (Sa^2]$, $a \in (a^2S])$.An ordered semigroup S is called intra-regular if for all $a \in S$ *there exists* $x, y \in S$ such that $a \leq xa^2y$ *i.e*, $a \in (Sa^2S]$. An ordered semigroup S is called a left (resp. right) duo if the right (resp. left) ideals are two sided. S is called duo if it is right duo and left duo.

A congruence ρ on S is called a semilattice congruence if for all $a, b \in S$ $a\ \rho\ a^2$ *and* $ab\ \rho\ ba$. A semilattice congruence ρ on S is called complete if $a \leq b$ implies $(a, ab) \in \rho$. Equivalently: An ordered semigroup S is called a (complete) semilattice of subsemigroup of type τ if there exists a (complete) semilattice congruence ρ on S such that $(x)_\rho$ is a type τ subsemigroup of S. Equivalently: There exists a semilattice Y and a family of subsemigroups $\{S\}_{\alpha \in Y}$ of type τ of S such that (i) $S_\alpha \cap S_\beta = \phi$ for any $\alpha,\ \beta \in\ Y$ *with* $\alpha \neq \beta$, (ii) $S = \bigcup_{\alpha\ \in\ Y}\ S_\alpha$, $(iii) S_\alpha S_\beta\ \subseteq\ S_{\alpha\ \beta}$ for any $\alpha,\ \beta \in\ Y$, and (iv) $S_\beta \cap (S_\alpha] \neq \phi$ implies $\beta\ \preceq\ \alpha$, where $\preceq$ is the order of the semilattice Y defined by $\preceq := \{(\alpha,\ \beta)\ \mid\ \alpha = \alpha\ \beta\ (\beta\ \alpha)\}$.

3 Scope of Work:

A semigroup S is uniformly $\pi-$regular if it is a semilattice of completely Archimedean semigroups. Studies of decompositions of semigroups into semilattices of completely Archimedean semigroups began in L. N. Shevrins papers. The final results of his several year long investigations are given in [9]. In ordered semigroup (with a compatible partial order) there are vast scope to study completely Archimedeaness. This notion have not been yet studied.

4 Plan of Work:

I want to investigate the structure of ordered semigroups as follows:

(1) Representing completely Archimedean ordered semigroups as union of if and only if S is π-regular and semilattice decompositions of these semigroups.

(2) Characterizing of left Archimedean ordered semigroups.

(3) Characterizing nil extensions of both completely Archimedean and left Archimedean ordered semigroups.

(4) Characterizing Chain decompositions of Archimedean (left) ordered semigroups.

References

[1] A. H. Clifford, Semigroups admitting relative inverses, *Annals of Math.*, **42**(1941), 1037-1049.

[2] G.Birkhoff,Lattice Theory, *American Math. Soc. coll. Publ., Providence*,**Vol XXV**(1967). **60**(2017), 246-252.

[3] J.L. Galbati and M.L. Veronesi, On quasi completely regular semigroup, *Semigroup forum* **29**(1984), 271-275.

[4] J.M. Howie, Fundamentals of Semigroup Theory, *Clarendon Press, Oxford*, **1**995.

[5] L. N. Shevrin, Theory of epigroups I, *Mat. Sb.* **185(8)** (1994), 129-160, (in Russian)

[6] N.Kehayopulu, Remark in ordered semigroups, *Math. Japonica* **35**(1990), 1061-1063.

[7] N.Kehayopulu, Ideals and Green's relations in ordered semigroups, *International Journal of Mathematics and Mathematical Sciences* , (2006), 1-8,Article ID 61286.

[8] S. Bogdanovic and S. Milic, A nil-extension of a completely simple semigroup, *Publ. Inst. Math.* **36(50)** (1984), 4550.

[9] Y. Cao, Chain decomposition of ordered semigroups, *Semigroup Forum*, **65**(2002), 83-106.